NONHOLONOMIC APPROACH TO ROTATING MATTER IN GENERAL RELATIVITY

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Rigidly rotating stationary matter in general relativity has been investigated by Kramer (Class. Quantum Grav. **2** L135 (1985)) by the Ernst coordinate method. A weakness of this approach is that the Ernst potential does not exist for differential rotation. We now generalize the techniques by the use of a nonholonomic and nonrigid frame. We apply these techniques for differentially rotating perfect fluids. We construct a complex analytic tensor, characterizing the class of matter states in which both the interior Schwarzschild and the Kerr solution are contained. We derive consistency relations for this class of perfect fluids. We investigate incompressible fluids characterized by these tensors.

1 Introduction

In this contribution, we briefly describe a new approach to rotating matter in general relativity with applications to differentially rotating perfect fluids. The full details of this method can be found in Ref. 1.

The essence of our treatment is the generalization of the Ernst-potential coordinate method to space-times in which the Ernst potential does not exist at all. (Examples of space-times where the Ernst potential does exist are the stationary axisymmetric vacuum and rigidly rotating perfect fluids.) The complex 1-form $\mathbf{G}$, introduced in Ref. 2 exists whenever a (non-null) Killing vector is given. By its use, we define a complex nonholonomic basis for axistationary space-times with the property that it becomes a natural basis whenever an Ernst potential exists. If the latter is not the case, the basis is noncommutative, *i.e.* the structure functions do not vanish.

The 3-space is conformally flat if the Cotton-York tensor $Y_i^\ell = \epsilon^{jk\ell}\left(R_{i[j;k]} - (1/4)g_{i[j}R_{;k]}\right)$ vanishes. The *Simon tensor*[3] is essentially an analytic continuation of the Cotton-York tensor in terms of the Ernst potential. Space-times like the Kerr metric, the interior Schwarzschild, Wahlquist[4] and the Kramer metrics[5] have a vanishing Simon tensor. We further generalize the Simon tensor for situations with no Ernst potential. Solving our field equations for perfect fluids with a vanishing complex tensor S_{ik}, we establish the following:

Theorem. *There are no incompressible perfect fluids with a vanishing* S_{ik} *tensor.*

2 Stationary perfect fluid space-times

We write the metric of a stationary space-time in terms of the metric g of the Killing trajectories:

$$\mathrm{d}s^2 = r(\mathrm{d}t + \omega_i \mathrm{d}x^i)^2 - \frac{1}{r} g_{ij} \mathrm{d}x^i \mathrm{d}x^j \ . \tag{1}$$

Introducing the 3-dimensional complex 1-form $\mathbf{G} \stackrel{\text{def}}{=} \left(\mathrm{d}r + \mathrm{i}r^2 \ {*\mathrm{d}\omega}\right)/(2r)$, the Einstein equations become (For our notation, *cf.* Ref. 1):

$$\begin{aligned} G^i{}_{;i} &= (\mathbf{G}\cdot\mathbf{G}) - (\mathbf{G}\cdot\bar{\mathbf{G}}) + kr^{-2} T^*_{oo}, & (2)\\ G_{i;j} - G_{j;i} &= \bar{G}_i G_j - G_i \bar{G}_j + \mathrm{i}kr^{-2}\epsilon_{ijk} T_o^{*k}, & (3)\\ R_{ij} &= -G_i\bar{G}_j - \bar{G}_i G_j - kr^{-2}\left(T^*_{ij} - g_{ij}T^*_{oo}\right), & (4) \end{aligned}$$

where $T^*_{\mu\nu} \stackrel{\text{def}}{=} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$. The energy-momentum tensor is $T_\mu^{\ \nu} = (\mu+p)u_\mu u^\nu - p\delta_\mu^\nu$ with the normalization condition for the 4-velocity $u_o^2 - g_{jk}u^j u^k = r$.

We introduce the complex nonholonomic basis for axistationary space-times $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{L})$, where $\mathbf{L} = \partial/\partial\varphi$ is the axial Killing vector, and the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ are defined by

$$\mathbf{G} = \frac{1}{2r}(\alpha\mathbf{e_1} + \beta\mathbf{e_2}), \quad \bar{\mathbf{G}} = \frac{1}{2r}(\beta\mathbf{e_1} + \gamma\mathbf{e_2}), \tag{5}$$

with $\alpha = 4r^2(\mathbf{G}\cdot\mathbf{G}),\ \beta = 4r^2(\mathbf{G}\cdot\bar{\mathbf{G}}),\ \gamma = \bar{\alpha}$. The metric reads

$$[g^{\mathbf{ik}}] = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \varrho^{-2} \end{bmatrix}, \tag{6}$$

where $\varrho^2 = (\mathbf{L}\cdot\mathbf{L})$. From the definition of $\mathbf{G}$ we have $r\left(\mathbf{G} + \bar{\mathbf{G}}\right) = \mathrm{d}r$, and $r_{,\mathbf{1}} \equiv \mathbf{e_1}r = r_{,\mathbf{2}} \equiv \mathbf{e_2}r = 1/2$. The *structure functions* $c^{\mathbf{i}}{}_{\mathbf{jk}}$ are defined by $[\mathbf{e_j}, \mathbf{e_k}] = c^{\mathbf{i}}{}_{\mathbf{jk}}\mathbf{e_i}$. The nonvanishing components are $c^{\mathbf{1}}{}_{\mathbf{12}} = -c^{\mathbf{2}}{}_{\mathbf{12}} = 2\mathrm{i}k\varrho T_o^{*\varphi}/(r\sqrt{D}) \stackrel{\text{def}}{=} \varepsilon$, where $D \stackrel{\text{def}}{=} \alpha\gamma - \beta^2 < 0$. The Einstein equation (2) takes the form

$$\alpha_{,\mathbf{1}} + \beta_{,\mathbf{2}} - \frac{\alpha}{r} + (\alpha\mathbf{e_1} + \beta\mathbf{e_2})\ln\left(\frac{\varrho}{\sqrt{D}}\right) - \frac{2k}{r}T^*_{oo} + (\alpha+\beta)\varepsilon = 0 \ , \tag{7}$$

which reduces to the Ernst equation in vacuum.

3 The complex tensor

We introduce the *complex* tensor

$$S_i^\ell = \epsilon^{jk\ell}\left\{2g_{ij}g^{rs}G_{[k;|r|}G_{s]} - 2G_{k;i}G_j - \mathrm{i}kr^{-2}\epsilon_{jk}{}^{r}G_{(i}T^*_{or)}\right\}, \tag{8}$$

which is symmetric and trace-free even in the presence of matter. In vacuo, S_i^ℓ equals the Simon tensor[3].

In the axisymmetric case the condition $S_i^\ell = 0$ gives two equations

$$\alpha_{,\mathbf{2}} = 0 \ , \quad (\alpha \mathbf{e_1} + \beta \mathbf{e_2}) \ln \varrho - \frac{\alpha}{2r} + \frac{1}{4}\alpha_{,\mathbf{1}} - \frac{k}{r} T^*_{oo} = 0. \tag{9}$$

Combining with Eq. (7), we obtain

$$D(\ln \varrho)_{,\mathbf{1}} + \frac{1}{2}\beta\gamma_{,\mathbf{2}} - \gamma\beta_{,\mathbf{2}} - D\varepsilon = 0 \ , \quad \alpha(\ln c)_{,\mathbf{1}} = F \ , \tag{10}$$

where $F \stackrel{\text{def}}{=} (k/r)T^*_{oo} - \beta\varepsilon$ and $c \stackrel{\text{def}}{=} D^{-1/2} r^{-1} (\alpha\gamma)^{3/4} \varrho$. We describe the matter by ε and F. By the normalization condition of u^μ,

$$D\varepsilon^2 + [2(F + \beta\varepsilon) + k(\mu - p)][2(F + \beta\varepsilon) - k(\mu + 3p)] = 0 \ . \tag{11}$$

The Bianchi identities give

$$kp_{,\mathbf{1}} = \frac{1}{2r}\left\{kp + \frac{1}{2}\varepsilon(\beta - \gamma) - F - \beta\varepsilon + [2r(F + \beta\varepsilon) - k(\mu + 3p)r](\ln \varrho)_{,\mathbf{1}}\right\}. \tag{12}$$

The remaining field equations are as follows,

$$\begin{aligned}
\alpha\alpha_{,\mathbf{11}} - \frac{3}{4}(\alpha_{,\mathbf{1}})^2 &= 4\alpha(2\varepsilon\beta_{,\mathbf{1}} + \beta\varepsilon_{,\mathbf{1}} + F_{,\mathbf{1}}) - 2(3\beta\varepsilon + 2F)\alpha_{,\mathbf{1}} \\
&\quad + 4F(2\beta\varepsilon + F) + \frac{4\alpha}{r}\left(\beta\varepsilon + r\gamma\varepsilon^2 + F - kp\right) && (13) \\
F_{,\mathbf{1}} + \varepsilon F &= \gamma\varepsilon^2 + \frac{\gamma}{2rD}\left[k(\mu + 3p)\beta - \left(\alpha\gamma + \beta^2\right)\varepsilon - 2F\beta\right] && (14) \\
\alpha F_{,\mathbf{1}} + (\alpha - \beta)\varepsilon F &= \alpha\varepsilon(\gamma - \beta)\left(\varepsilon - \frac{1}{2r}\right) - \frac{3}{4}\beta\varepsilon\alpha_{,\mathbf{1}} - \alpha\beta\varepsilon_{,\mathbf{1}} \ . && (15)
\end{aligned}$$

In general, the first order equations can be solved for the $\mathbf{e_1}$ derivatives of all our functions except for the fluid energy density μ. For an incompressible fluid the second order equation (13) together with the integrability conditions of our variables gives a system of algebraic equations. The proof of our Theorem follows by showing the inconsistency of these equations. If μ is not constant, one can still obtain equations linear in $\mu_{,\mathbf{1}}$ and $\mu_{,\mathbf{2}}$. At this time we do not know if the resulting algebraic equations yield any other solution than those found by Wahlquist[4] and Kramer[5].

References

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